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WITTGENSTEIN'S LECTURES
on the Foundations of Mathematics
Cambridge, 1939

FROM THE NOTES OF
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rule is a certain expression, and then there is a certain technique of applying this rule. How is this technique given? Either in examples or not. Then what is the right application? Could it be the natural step and not the convincing step?

Suppose that I tell you to multiply 418 by 563. Do you *decide* how to apply the rule for multiplication? No; you just multiply. Probably no rule at all would come into your head. And if one did, no other rule for the application of the first rule would come into your head. It is not a decision; nor is it an intuition.

Malcolm: Couldn't we state the rules in a general way? For instance, we might say "3 X 3 is *always* 9."

Wittgenstein: We might say that, but in some higher multiplication give an exception. If you read the newspapers and see how people get round pacts, you should not be surprised at this.

We might explain it in several ways. We might, for instance, say, "Always, except in this one case." Or "Yes, 3 X 3 is always 9, but in this case write it as 7." Or we might find the use of "always" unnatural and give it up.—But the new multiplication is a new rule.

Russell says, roughly: "After all, it is not self-evidence which must guide one in the choice of primitive propositions. On the contrary, one is guided sometimes by the results which a given choice produces. Many primitive propositions are shown to be true by what follows from them." [3]—You may choose them because you want to get to a certain point. Not because they are indubitable.

XXV

The idea that there are two kinds of proof: 'the *real* proof'—the proof which gives a firm ground to the proposition, so that it is

3. Cf. *Principia Mathematica* (Cambridge, 1910), I, 13.

unshakeable and won't fall—and the proof that is to convince you. It doesn't make the proposition unshakeable—it only makes you believe that it is unshakeable.

This idea comes from a false view of what a proof actually does—and a false idea of the role which mathematical and logical propositions play.

Consider Professor Hardy's article ("Mathematical Proof") and his remark that "to mathematical propositions there corresponds—in some sense, however sophisticated—a reality".[1] (The fact that he said it does not matter; what is important is that it is a thing which lots of people would like to say.)

Taken literally, this seems to mean nothing at all—*what* reality? I don't know what this means.—But it is obvious what Hardy compares mathematical propositions with: namely physics.

Suppose we said first, "Mathematical propositions can be true or false." The only clear thing about this would be that we affirm some mathematical propositions and deny others. If we then translate the words "It is true . . ." by "A reality corresponds to . . ."—then to say a reality corresponds to them would say only that we affirm some mathematical propositions and deny others. We also affirm and deny propositions about physical objects.—But this is plainly not Hardy's point. If this is all that is meant by saying that a reality corresponds to mathematical propositions, it would come to saying nothing at all, a mere truism: if we leave out the question of *how* it corresponds, or in what sense it corresponds.

We have here a thing which constantly happens. The words in our language have all sorts of uses; some very ordinary uses which come into one's mind immediately, and then again they have uses which are more and more remote. For instance, if I say the word 'picture', you would think first and foremost of something drawn or painted and, say, hung up on the wall. You would not think of Mercator's projection of the globe; still less of the sense in which a man's handwriting is a picture of his character. A word has one or more nuclei of uses which come into every-

1. Page 4. "[Mathematical theorems] are, in one sense or another, however elusive and sophisticated that sense may be, theorems concerning reality . . ." Hardy does not speak of a correspondence to reality.

body's mind first; so that if one says so-and-so is also a picture—a map, or *Darstellung* in mathematics—in this lies a comparison: as it were, "Look at this as a continuation of that."

So if you forget where the expression "a reality corresponds to" is really at home—

What is "reality"? We think of "reality" as something we can *point* to. It is *this, that.*

Professor Hardy is comparing mathematical propositions to propositions of physics. This comparison is extremely misleading.

"To mathematical propositions there corresponds a reality" —if you take this in the sense of "Some mathematical propositions we affirm", then it is harmless but meaningless.

Or to say this may mean: these propositions are *responsible* to a reality. That is, you can't say just anything in mathematics, because there's the reality. This comes from saying that propositions of physics are responsible to that apparatus—you can't say any damned thing.

It is almost like saying, "Mathematical propositions don't correspond to *moods;* you can't say one thing now and one thing then." Or again it's something like saying, "Please don't think of mathematics as something vague which goes on in the mind." Because that has been said. Someone may say that logic is a part of psychology: logic treats of laws of thought and psychology deals with thought. You could get to the idea of logic as extremely vague, as psychology is so extremely vague. And if you oppose this you are inclined to say "a reality corresponds".

If you were to point out what mathematics is responsible to, then you would get the reality to which mathematics, in a sense, does correspond.

Here we see two kinds of responsibility. One may be called "mathematical responsibility": the sense in which one proposition is responsible to another. Given certain principles and laws of deduction, you can say certain things and not others.—But it is a totally different thing if we ask, "And now what's *all* this responsible to?" The axioms and the way of drawing conclusions may be said to be responsible to something, or not to be arbitrary.

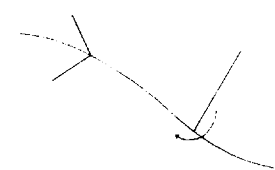
When we speak of the responsibility of one proposition to axioms and laws of transformation, I have constantly stressed that given a set of axioms and rules, we could imagine different ways of using them. You might say, "So, Wittgenstein, you seem to say there *is* no such thing as this proposition necessarily following from that."—Should we say: Because we point out that whatever rules and axioms you give, you can still apply them in ever so many ways—that this in some way undermines mathematical necessity?

Von Wright: We oughtn't to say that; for the kind of thing we get in mathematics is what we call necessity.

Wittgenstein: Yes, one answer is: "But this is what we *call* necessity. We say '25 X 25 = 625 follows necessarily from so-and-so.' "

Or if a person says that this undermines mathematical necessity, you might ask, "What is your paradigm of necessity?—Show me first of all what you call necessity, and then we'll talk about whether this is necessity."

Now we have various paradigms in this case. One is *regularity.* Another: we say, "It's necessary that he'll come here"—we cannot get on without him; or something nasty will happen if he doesn't. So here if we say a thing is necessary, there must be something that goes wrong if it doesn't happen.—Or we might have a game in which some moves are necessary and not others.



"*Here* the rules say you must turn right; *here* you may go whichever way you like."

What is necessary is determined by the rules.—We might then ask, "Was it necessary or arbitrary to give these rules?" And here we might say that a rule was arbitrary if we made it just for fun and necessary if having this particular rule were a matter of life and death.

We must distinguish between a necessity in the system and a necessity of the whole system. This is the point of von Wright's remark just now, that this is what we *call* necessary. He might have said that in this case it is not a question of whether the system as a whole is necessary.

We have to distinguish between different senses of 'necessary'. If we teach a calculus—and we have to multiply 21 X 14—we say the answer necessarily follows from certain axioms or premises. The question to ask is: Necessarily as opposed to what? Presumably, as opposed to the case where in our practice we leave it open what follows—or else it is a pleonasm.

This is analogous to an ethical discussion of free will. We have an idea of compulsion. If a policeman grabs me and shoves me through the door, we say I am compelled. But if I walk up and down here, we say I move freely. But it is objected: "If you knew all the laws of nature, and could observe all the particles etc., you would no longer say you were moving freely; you would see that a man just cannot do anything else."—But in the first place, this is not how we use the expression "he can't do anything else". Although it is *conceivable* that if we had a mechanism which would show all this, we would change our terminology—and say, "He's as much compelled as if a policeman shoved him." We'd give up this distinction then; and if we did, I would be very sorry.

To say "If you multiply these two, you necessarily get such-and-such a number", if it means anything *at all*, must be opposed to a case where there is no necessity. Or else it's a pleonasm to say you *necessarily* get this—why not simply say that you *get* it? —We might speak of getting something but not necessarily, in the case of a calculus in which you could get more than one answer.

With regard to "responsibility to reality": On the one hand you might say, "This conclusion is responsible to certain axioms and certain rules." This responsibility is based on our peculiar practice of using these rules. But then there is another question: as to whether such a system as a whole is responsible to anything. And to investigate this I tried to point out what *does* go wrong if we draw conclusions in a different way. We saw two things.

(1) We are then no longer inclined to use words as we do use them: for example, to use a certain word as negation—"there's no such thing"—and a certain word as a conjunction—"there's sugar and coal there".

If we give a word one particular partial use, then we are inclined to go on using it in one particular way and not in another. "Not" could be explained by saying such things as "There's not a penny here" or saying to a child "Must *not* have sugar" (preventing him). We haven't said everything but we have laid down *part* of the practice. Once this is done, we are inclined when we go on to adopt one step and not another—for example, double negation being equivalent to affirmation.

If the logical laws do not hold—we don't get the game we want to get, we don't play the game we want to play.

Suppose I said, "If you give different logical laws, you are giving the words the wrong meaning." This sounds absurd. What is the wrong meaning? Can a meaning be wrong? There's only one thing that can be wrong with the meaning of a word, and that is that it is unnatural. To give "not" the meaning of "and" and vice versa is not at all unnatural. But there are other things which are unnatural. For instance, we said we don't want to say "reddish-green". It is unnatural—unnatural for *us*—to use "red" and "green" in the way we're accustomed, and then to go on to talk of "reddish-green". And it is unnatural for us, though not for everyone in the world, to count: "one, two, three, four, five, many". We just don't go on in that way.

(2) If we allow contradictions in such a way that we accept that *anything* follows, then we would no longer get a calculus, or we'd get a useless thing resembling a calculus.

Suppose you had to say to what reality this—"There is no reddish-green"—is responsible. Where is the reality corresponding to the proposition "There is no reddish-green"? (This is entirely parallel to Hardy's "reality".)—It makes it look the same as "In this room there is nothing yellowish-green." This is of practically the same appearance—but its use is as different as hell.

If we say there's a reality corresponding to "There is no reddish-green", this immediately suggests the kind of reality corresponding to the other proposition. Which reality would you say corresponds to that? We have in mind that it must be a reality roughly of the sort: the absence of anything which has this colour (though that is queer, because, in saying that, we are saying just the same thing over again). It is superhuman not to think of the reality as being something similar in the case of "There is no reddish-green".

Now there *is* a reality corresponding to this, but it is of an *entirely* different sort. One reality is that if I had arranged for myself to call something reddish-green, other people would not know what to say. (Although I might appeal to examples in support of my use: to certain holly leaves which are red at one point and green at another and at a point in between they are a sort of irridescent black. I've often thought that if I had to call something reddish-green it would be that.)

"This is a flimsy thing to consider—whether one is *inclined* to say this or that." But it is no more flimsy than whether one is inclined to compare it with one thing or another thing, or whether one is inclined to use this picture or that picture.

I once knew a boy who talked of the 'dark notes' on the piano, not meaning the black notes but the low notes, although he had never heard them referred to as dark.[2] We might say, "He felt a similarity between darkness and low notes." If someone asked, "What *is* this similarity he felt? Where does it lie?", what could you say?—This *is* the similarity: that he wanted to say "dark".

Isn't this what we call "noticing a similarity"? If we say, "He is inclined to use the word 'dark' ", this is like "He is inclined to use this picture ■ instead of that □."

This [inclination to call the low notes 'dark'] may be connected with all sorts of facts: that a child is frightened in the dark, not in the light; that he knows what a growl is, and is more inclined to be frightened by deep rumbling than by twittering.

What connexions we are inclined to make is (a) of the most enormous importance, (b) hangs together with all sorts of things.

If you were to say what reality corresponds to "There is no reddish-green"—I'd say: You may say a reality corresponds, only (1) it is of an entirely different kind from what you assume; (2) [what you have is a *rule*,] namely the [rule] that this expression can't be applied to anything. The correspondence is between this

2. Cf. *The Brown Book*, Part II, §4; *Eine Philosophische Betrachtung*, §§111, 123–124, in *Schriften* 5 (Frankfurt, 1970).

rule and such facts as that we do not normally make a black by mixing a red and a green; that if you mix red and green you get a colour which is 'dirty', and dirty colours are difficult to remember. All sorts of facts, psychological and otherwise.

We might ask, "Does any reality correspond to: 'A double negation gives an affirmation'?"

Think of "Two such-and-such things give such-and-such." Like: "If you turn this round twice . . ." "If you insert two pennies in the slot, so-and-so will happen." "Two turns of the handle produce such-and-such an effect."

We might think of two cases here. We might think of a light switch: turning it around once turns on the light, turning it around again turns it off. Or we might think of turning a match through 180°, then 180° again, so that the head faces the original direction.—Now there is a great difference between these two illustrations. The case of the match might be called a geometrical demonstration; the other might be called an experiment—to see how the switch works.—To put this in another way: the light might fail to work. But what would correspond to this in the case of the match? Nothing geometrical; what would correspond to it would be, say, the match breaking.

(Suppose someone says, "That space is three-dimensional is a matter of experience." What experiments would be made? Should we hold up three sticks at right angles and say, "Obviously we can't put another stick in at right angles to these"? What rot!)

Now suppose I turn the match around. What reality did I point to? What did I show you?

I might show you that the light switch goes on at every other turn, and not, say, at every third turn. But if you say I showed you that turning the match through 180° twice brings it back to the same position—isn't this just a matter of definition?

You could have a case of measuring. You might take a protractor, measure off 180°; you measure off 180° *again*, and turn it, and see whether it points in the same direction as before. This would be an experiment.—If it didn't point in the same direction, would you say the protractor was wrong, that it had expanded, etc.,—or would you say that in this case twice 180° does not bring you back to the same position?—So if you hold out the match and turn it round, if you say you are 'demonstrating something' —I don't know what you're demonstrating. You're turning a match.

"What reality corresponds to the proposition that if you turn a match twice through 180° it gets back to its original position?"

If this is a geometrical proposition, the reality which corresponds is: if we use a good protractor, then normally it brings us back, or more nearly back the better it is (where 'better' is determined by other criteria).

Yes—a reality corresponds to it, but it isn't of the kind you at first expect. At first you imagine that this is an experiment, like the experiment with the light switch. Then you discover that there is a reality corresponding to it but—if I may use the phrase—a much less clear reality: all sorts of things about protractors, etc., the fact that we can normally turn this round, and so on.

What you are saying is not an experiential proposition at all, though it sounds like one; it is a *rule*. That rule is made important and justified by reality—by a lot of most important things.

If you say, "Some reality corresponds to the mathematical proposition that $21 \times 14 = 294$", then I would say: Yes, reality, in the sense of experiential (empirical) reality *does* correspond to this. For example, the central reality that we have methods of representing this so that it can all be seen at a glance. In such a case as 21×14 nothing is easier than to lay out 21 rows of 14 matches and then count them; and then there is no doubt at all that *all of us* would get the same result. This is an experiential result; and it is immensely important. In such a case, if we looked at the things, we could easily notice if a thing vanished. We would all immediately agree that something had vanished, or that nothing had vanished.

I want to talk about the question whether one can justify the results of mathematical calculations by means of Russell's logic,

or whether they depend upon certain quite different techniques; say the technique of being able to compare two numbers of objects in a certain way. For instance this: *Principia Mathematica* has been printed in a few thousand copies. We say they all contain the same proofs. There is a way in which these copies have been produced, and this has been checked; and this satisfies us.

XXVI

If one talks about a reality corresponding to mathematical propositions and examines what that might mean, one can distinguish two very different things:

(1) If we talk of an experiential proposition, we may say a reality corresponds to it, if it is true and we can assert it.

(2) We may say that a reality corresponds to a *word*, say the word "rain"—but then we mean something quite different. This word is used in "it rains", which may be true or false; and also in "it doesn't rain". And in this latter case if we say "some phenomenon corresponds to it", this is queer. But you might still say something corresponds to it; only then you have to distinguish the sense of "corresponds".

If you say, "Something corresponds to the word 'red', namely this colour"—how does it correspond if you say (truly), "There's nothing red in this room"? And you might also have "There's nothing red in the world."

Von Wright: It doesn't seem to me there is a very big difference between correspondence in the case of a sentence and correspondence in the case of a word.

Wittgenstein: There is an enormous difference.—Suppose I spoke of the reality corresponding to this sentence. I may mean two entirely different things: (a) I might mean that the sentence is true; (b) I might mean that there is a reality corresponding to the words in it—that is, that the sentence has a meaning. And these two things are entirely different. In the one case, by saying "A reality corresponds to so-and-so" we are affirming a sentence.